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On the theory of algebraic invariants of vector spaces of Killing tensors

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Abstract

We employ a hybrid version of the classical Cartan method of moving frames to develop a practical algorithm for the generation of isometry group invariants and covariants for arbitrary vector spaces of Killing tensors defined on any flat pseudo-Riemannian manifold. We then apply our algorithm to construct a set of fundamental covariants for the space of valence-two Killing tensors defined in three-dimensional Euclidean space and use them to invariantly characterize the associated eleven orthogonally separable webs.

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1. Introduction

Killing tensors and their associated orthogonally separable webs play a pivotal role in the Hamilton–Jacobi theory of separation of variables and in the integrability of finite-dimensional Hamiltonian systems [2]. These classical problems arising from mathematical physics can be studied from the viewpoint of the *invariant theory of Killing tensors* (ITKT), a synergy of the classical invariant theory (CIT) of polynomials with the study of Killing tensors defined in pseudo-Riemannian manifolds of constant curvature. In this paper, we revisit the problem originally treated in [12] of computing isometry group invariants for the space of Killing tensors defined in three-dimensional Euclidean space $\mathbb{E}^3$ and the solution of the equivalence problem for the associated eleven orthogonally separable webs. We accomplish these goals by presenting an apparently novel method for the generation of group invariants and covariants. Furthermore, our method is extended to spaces of Killing tensors defined on *arbitrary* flat manifolds independent of dimension and signature. We shall demonstrate that deriving invariants in ITKT is no longer a computational hurdle that it once was thought to be.

In the last few years, we have witnessed a steady development of the invariant theory of Killing tensors. This continues to flourish at a rapid pace today. The computation of fundamental sets of isometry group invariants has been

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accomplished for a wide variety of vector spaces of Killing tensors [1,4,5,12,14,17–21,25,26,32]. The classification of (orthogonally) separable webs on the Euclidean and Minkowski planes, $\mathbb{E}^2$ and $\mathbb{M}^2$, and in three-dimensional Euclidean space $\mathbb{E}^3$, using various invariant based methods, is also well known [4,12,17,19,20,25]. Of particular significance is the $\mathbb{E}^3$ case [12] which entails the study of a twenty-dimensional vector space. It is the highest-dimensional vector space considered to date in the context of ITKT.

The invariant classification of separable webs in $\mathbb{E}^3$, as developed in the paper of Horwood, McLenaghan and Smirnov [12], was a substantial breakthrough for ITKT. The theory of the present paper now permits us to streamline the results in [12] and realize significant improvements in two ways. Firstly, we can generate invariants "by hand" and they can be expressed in an efficient compact notation, whereas the derivation of the group invariants in [12] relied heavily on computer algebra and the resulting invariants did not appear to enjoy any obvious structure. Our new algorithm even extends to arbitrary valence, dimension and signature. Secondly, we can now give an invariant classification of the eleven separable webs in $\mathbb{E}^3$ based purely on isometry group covariants and can demonstrate that our procedure for classifying separable webs is generally applicable to $\mathbb{E}^n$. In [12], the invariant characterization was not possible using invariants alone, unlike the simpler problem on $\mathbb{E}^2$ [4,17,19,25]. Instead, the authors in [12] required reduced invariants, i.e. invariants defined on invariant subspaces, in order to complete the classification.

The mathematics employed in ITKT has its roots in Cartan's philosophy of geometry [1,9] and can thus be explained in the elegant and modern language of Lie groups, homogeneous spaces, fibre bundles and moving frames. In particular, the derivation of the group invariants and covariants uses the classical moving frame method [7,8] in conjunction with the recursive construction of the moving frame introduced in [15]. The necessity of covariants in ITKT, in addition to pure invariants alone, comes as no surprise, for in the context of CIT, Olver states:

> *"While invariants are of fundamental importance ...*
> *by themselves they do not paint the entire picture."* [22, pg. 25]

The classification of separable webs in $\mathbb{E}^3$ is a prime example of when invariants alone fail to "paint the entire picture". The group orbits of the associated characteristic Killing tensors characterizing the eleven webs do not all have the same dimension; those Killing tensors admitting a translational or rotational symmetry have orbits of non-maximal dimension. Thus, one can only expect a set of invariants to give at best a *local* characterization of the orbits. This lack of "discriminating power" in the pure invariants can be improved by considering *covariants* which are essentially invariants also having explicit dependence on the pseudo-Cartesian coordinates. The use of covariants is a manifestation of *prolongation*, which effectively fixes many of the deficiencies in the group action by extending the space on which it acts [3]. In this paper, we shall see that covariants serve as a useful tool for characterizing group orbits in ITKT and completely solve the classification problem of the separable webs in $\mathbb{E}^3$.

With the inception of this paper into the ITKT literature, we now have a practical algorithm for generating invariants, covariants and joint invariants for vector spaces of Killing tensors, *independent* of the dimension and signature of the manifold over which the space is defined and *independent* of the valence of the tensors. The algorithm should be contrasted with previous approaches to computing invariants in ITKT which were largely dimensionally dependent. Moreover, our algorithm leads to a classification of the eleven separable webs in $\mathbb{E}^3$ based on *covariants alone*. These advancements to ITKT also prove useful in the solution of the equivalence problem in three-dimensional Minkowski space $\mathbb{M}^3$, a manifold which admits thirty-nine distinct separable webs [11,13]. Finally, we remark that our method is equally applicable to CIT and, in fact, the algorithm of the present paper is an extension of the one developed in [10], thus further strengthening the synergy between CIT and ITKT.

The plan of the paper is as follows. In Section 2, we give an overview of ITKT culminating in the derivation of a representation of the isometry group on the vector space of Killing tensors of valence p defined on a general n-dimensional pseudo-Euclidean space. In Section 3, we present our algorithm for the generation of group invariants and covariants which is obtained from the representation derived in the previous section in conjunction with the application of the moving frame method. Finally, we begin Section 4 with a discussion of Cartan's geometry and its extension to ITKT and then give a classification of the separable webs in $\mathbb{E}^3$ using a set of covariants generated by the algorithm in Section 3.

2. Representations of the pseudo-Euclidean groups

A representation of the Lie group which acts on a space of Killing tensors is pivotal in the determination of group invariants. We now outline the key elements of ITKT beginning with the definition of a Killing tensor and culminating in an explicit form of the representation of the isometry group of n-dimensional pseudo-Euclidean space. In what follows, the notation $(\mathcal{M}, \boldsymbol{g})$ denotes a real n-dimensional pseudo-Riemannian manifold of constant curvature with covariant metric tensor $\boldsymbol{g}$.

Definition 1. A Killing tensor $\boldsymbol{K}$ of valence p defined in $(\mathcal{M}, \boldsymbol{g})$ is a symmetric contravariant tensor of valence p satisfying the Killing tensor equation

$$[\boldsymbol{K}, \boldsymbol{g}^{-1}] = 0, \tag{2.1}$$

where [,] denotes the Schouten bracket [23] and $\boldsymbol{g}^{-1}$ denotes the contravariant metric tensor.

We remark that in the case $p = 1$, $\boldsymbol{K}$ is a *Killing vector* and Eq. (2.1) reduces to $\mathcal{L}_{\boldsymbol{K}}\boldsymbol{g} = 0$, where $\mathcal{L}$ is the Lie derivative operator. In local coordinates, Eq. (2.1) becomes

$$g^{j(k}\nabla_j K^{i_1\cdots i_p)} = 0, \tag{2.2}$$

where ∇ is the Levi-Civita connection on $\mathcal{M}$ with respect to the metric $\boldsymbol{g}$.

It is straightforward to observe from Definition 1 that the set of all Killing tensors of valence p defined in $(\mathcal{M}, \boldsymbol{g})$, denoted by $\mathcal{K}^p(\mathcal{M})$, defines a real vector space. Its dimension d is derived independently in Delong [6], Takeuchi [29] and Thompson [30] and is given by

$$d = \dim \mathcal{K}^p(\mathcal{M}) = \frac{1}{n}\binom{n+p}{p+1}\binom{n+p-1}{p}. \tag{2.3}$$

Therefore, with respect to an appropriate basis, the general Killing tensor of $\mathcal{K}^p(\mathcal{M})$ is represented by d arbitrary parameters $a_1, \ldots, a_d$.

The (orientation preserving) isometry group $I(\mathcal{M})$ acts naturally on a space of Killing tensors. Indeed, by the push forward map, each element $h \in I(\mathcal{M})$ induces a non-singular linear transformation $\rho(h)$ of $\mathcal{K}^p(\mathcal{M})$ and, as established in [16, Theorem 3.5], the map

$$\rho : I(\mathcal{M}) \to \mathrm{GL}(\mathcal{K}^p(\mathcal{M})) \tag{2.4}$$

defines a representation of $I(\mathcal{M})$. Using the general Killing tensor $\boldsymbol{K}$ of $\mathcal{K}^p(\mathcal{M})$, one can compute the group action $I(\mathcal{M}) \circlearrowright \mathcal{K}^p(\mathcal{M})$ in terms of the parameters $a_1, \ldots, a_d$, thereby obtaining the explicit form of the transformation $h \cdot \boldsymbol{K} \equiv \rho(h)\boldsymbol{K}$. An invariant is thus any smooth real-valued function of $\mathcal{K}^p(\mathcal{M})$ invariant under the isometry group. A more precise definition is as follows.

Definition 2. Let $(\mathcal{M}, \boldsymbol{g})$ be a pseudo-Riemannian manifold of constant curvature and $p \geqslant 1$ be fixed. A smooth function $\mathcal{I} : \mathcal{K}^p(\mathcal{M}) \to \mathbb{R}$ is said to be an $I(\mathcal{M})$-invariant of $\mathcal{K}^p(\mathcal{M})$ iff it satisfies the condition

$$\mathcal{I}(h \cdot \boldsymbol{K}) = \mathcal{I}(\boldsymbol{K}) \tag{2.5}$$

for all $\boldsymbol{K} \in \mathcal{K}^p(\mathcal{M})$ and for all $h \in I(\mathcal{M})$.

The considerations thus far naturally extend to the study of *covariants* and *joint invariants* of vector spaces of Killing tensors, first introduced in the context of ITKT by Smirnov and Yue [25]. An $I(\mathcal{M})$-covariant of $\mathcal{K}^p(\mathcal{M})$ is simply an $I(\mathcal{M})$-invariant of the extended or prolonged space $\mathcal{K}^p(\mathcal{M}) \times \mathcal{M}$, while a joint invariant is an $I(\mathcal{M})$-invariant of the q-fold product space $\mathcal{K}^{p_1}(\mathcal{M}) \times \cdots \times \mathcal{K}^{p_q}(\mathcal{M})$.

For the remainder of this paper, we shall restrict our attention to the case when $\mathcal{M}$ is $\mathbb{E}^{n-s,s}$, the n-dimensional flat manifold $\mathbb{R}^n$ equipped with the pseudo-Euclidean metric

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = -(\mathrm{d}x^1)^2 - \cdots - (\mathrm{d}x^s)^2 + (\mathrm{d}x^{s+1})^2 + \cdots + (\mathrm{d}x^n)^2, \tag{2.6}$$

for a fixed signature s. The isometry group of $\mathbb{E}^{n-s,s}$ is the special pseudo-Euclidean group $\mathrm{SE}(n-s,s) = \mathrm{SO}(n-s,s) \ltimes \mathbb{R}^n$, where $\mathrm{SO}(n-s,s)$ is the Lie group of pseudo-orthogonal rotations with positive unit determinant.

In particular, $\mathrm{SE}(n) \equiv \mathrm{SE}(n,0)$ is the familiar group of rigid motions (rotations and translations) acting in Euclidean space $\mathbb{E}^n \equiv \mathbb{E}^{n,0}$. If $s = 1$, $\mathbb{M}^n \equiv \mathbb{E}^{n-1,1}$ is an n-dimensional Minkowski space and $\mathrm{SO}(n-1,1)$ is the special Lorentz group.

A primary goal of this paper is to present an algorithm which efficiently generates $\mathrm{SE}(n-s,s)$-invariants and covariants of $\mathcal{K}^p(\mathbb{E}^{n-s,s})$, for any p, n and s. This problem, together with the associated problem of classifying the orbits of $\mathcal{K}^p(\mathbb{E}^{n-s,s})$ (and subsets thereof), has been treated extensively in the ITKT literature for specific cases of p, n and s. On the Euclidean and Minkowski planes, $\mathbb{E}^2$ and $\mathbb{M}^2$, invariants and classification schemes for $\mathcal{K}^2(\mathbb{E}^2)$ and $\mathcal{K}^2(\mathbb{M}^2)$ have been completed using a variety of methods (see [4,17,19,25] and [4,18,20,25,26], respectively). For higher valence cases, the isometry group invariants of $\mathcal{K}^3(\mathbb{E}^2)$ derived in [14] were, for the first time in the ITKT literature, presented in a compact tensorial notation. Yue [32] succeeded even further by computing invariants and covariants of $\mathcal{K}^p(\mathbb{M}^2)$ for *arbitrary* valence p. The study of $\mathcal{K}^2(\mathbb{E}^3)$ in [12] also included the presentation of invariants in a compact form. This paper marked the first time such a result was presented on a vector space of Killing tensors defined in a three-dimensional space. The representation of SE(3) on $\mathcal{K}^2(\mathbb{E}^3)$ motivates several ideas used in the derivation of the general case and so it is useful to review them now.

It is well known that any Killing tensor defined on a manifold of constant curvature is expressible as a sum of symmetrized products of Killing vectors [6,29,30]. Summarizing the results in [12], it follows that a basis for the Lie algebra of Killing vectors in $\mathbb{E}^3$ may be written in Cartesian coordinates x^i according to

$$\boldsymbol{X}_i = \frac{\partial}{\partial x^i}, \qquad \boldsymbol{R}_i = \epsilon^k{}_{ji} x^j \boldsymbol{X}_k, \tag{2.7}$$

for $i = 1, 2, 3$, where ϵ_{ijk} is the Levi-Civita tensor. In (2.7) and throughout the paper, we are using the summation convention and lowering and raising tensor indices with the metric g_{ij} and its inverse g^{ij}. Therefore, the general Killing tensor in $\mathcal{K}^2(\mathbb{E}^3)$ may be expressed as

$$\boldsymbol{K} = A^{ij}\boldsymbol{X}_i \odot \boldsymbol{X}_j + 2B^{ij}\boldsymbol{X}_i \odot \boldsymbol{R}_j + C^{ij}\boldsymbol{R}_i \odot \boldsymbol{R}_j, \tag{2.8}$$

where A^{ij}, B^{ij} and C^{ij} are constant 3×3 matrices and satisfy the symmetry properties $A^{ij} = A^{(ij)}$ and $C^{ij} = C^{(ij)}$. The action $\mathrm{SE}(3) \circlearrowright \mathbb{E}^3$ is given by

$$x^i = \lambda^i{}_j \tilde{x}^j + \delta^i, \tag{2.9}$$

where $\lambda^i{}_j \in \mathrm{SO}(3)$, $\delta^i \in \mathbb{R}^3$ and $\tilde{x}^i$ denote the transformed set of Cartesian coordinates. Using (2.9), it follows that the Killing vectors (2.7) transform according to

$$\boldsymbol{X}_i = \lambda_i{}^j \tilde{\boldsymbol{X}}_j, \qquad \boldsymbol{R}_i = \lambda_i{}^j \tilde{\boldsymbol{R}}_j + \mu_i{}^j \tilde{\boldsymbol{X}}_j, \tag{2.10}$$

where

$$\mu_i{}^j = \epsilon^k{}_{\ell i} \lambda_k{}^j \delta^\ell. \tag{2.11}$$

The group action $\mathrm{SE}(3) \circlearrowright \mathcal{K}^2(\mathbb{E}^3)$ can now be derived using the transformation rules (2.10) in conjunction with (2.8) leading to

$$\begin{aligned}
\tilde{A}^{ij} &= \lambda_k{}^i \lambda_\ell{}^j A^{k\ell} + 2\lambda_k{}^{(i} \mu_\ell{}^{j)} B^{k\ell} + \mu_k{}^i \mu_\ell{}^j C^{k\ell},\\
\tilde{B}^{ij} &= \lambda_k{}^i \lambda_\ell{}^j B^{k\ell} + \mu_k{}^i \lambda_\ell{}^j C^{k\ell},\\
\tilde{C}^{ij} &= \lambda_k{}^i \lambda_\ell{}^j C^{k\ell}.
\end{aligned} \tag{2.12}$$

Remark. The reader will notice that a total of twenty-one parameters are contained in the matrices A^{ij}, B^{ij} and C^{ij} which appears to disagree with Eq. (2.3) predicting that $\dim \mathcal{K}^2(\mathbb{E}^3) = 20$. The reason for this difference is because the twenty-one symmetrized products of Killing vectors in (2.8) are not linearly independent and admit a *syzygy* $g^{ij}\boldsymbol{X}_i \odot \boldsymbol{R}_j = 0$. A consequence of this syzygy is the invariance of the quantity $B^i{}_i = g^{ij}B_{ij}$, which follows directly from (2.12). Without loss of generality, we set this trace to zero, thereby reconciling the discrepancy. In the general case which will follow, a more complicated system of syzygies exist amongst the symmetrized products. In the derivation of the group action and the group invariants, they can be safely ignored. Indeed, once a full set of invariants are computed for a particular vector space of Killing tensors, quantities analogous to $B^i{}_i$ will naturally appear in the set and can thus be identified as constraints and not "true" group invariants.

There are two key elements used in the derivation of the group action (2.12) which prove paramount if one wishes to generate invariants in an efficient compact notation. Firstly, the basis of Killing vectors (2.7) is written covariantly; no preference is given to a particular coordinate or direction. Secondly, we do not assume an explicit representation of the isometry group. In other words, we do not explicitly parametrize the rotation (e.g. using the three Euler angles) because doing so might "disturb the symmetry" and the simplicity of the group action. In the derivation of (2.12), it is enough to assume that $\lambda^i{}_j$ satisfies the elementary properties of SO(3).

We now generalize these crucial ideas and derive the action $\mathrm{SE}(n-s,s) \circlearrowright \mathcal{K}^p(\mathbb{E}^{n-s,s})$ for general n, s and p. In order to express the Killing vectors of $\mathbb{E}^{n-s,s}$ in a tensorial form, we define a type $(1,3)$ tensor $\delta^\ell{}_{ijk}$ in terms of the Kronecker delta $\delta^i{}_j$ and the metric tensor g_{ij} according to

$$\delta^\ell{}_{ijk} = \delta^\ell{}_i g_{jk} - \delta^\ell{}_j g_{ik}. \tag{2.13}$$

We note that this tensor is related to the familiar generalized Kronecker delta $\bar{\delta}^{ij}_{k\ell} = \delta^i{}_k \delta^j{}_\ell - \delta^i{}_\ell \delta^j{}_k$ (see for example [28]) viz $\bar{\delta}^{ij}_{k\ell} = -g^{im}\delta^j{}_{k\ell m}$. With respect to a system of Cartesian coordinates x^i, we define the vector fields

$$\boldsymbol{X}_i = \frac{\partial}{\partial x^i}, \qquad \boldsymbol{R}_{ij} = \delta^\ell{}_{ijk} x^k \boldsymbol{X}_\ell, \tag{2.14}$$

for $i = 1, \ldots, n$ and $1 \leqslant i < j \leqslant n$. It follows that the $\frac{1}{2}n(n+1)$ vectors in (2.14) are indeed Killing vectors and form a basis for $\mathcal{K}^1(\mathbb{E}^{n-s,s})$. The action $\mathrm{SE}(n-s,s) \circlearrowright \mathbb{E}^{n-s,s}$ is given by (2.9) where $\lambda^i{}_j \in \mathrm{SO}(n-s,s)$ and $\delta^i \in \mathbb{R}^n$. It is straightforward to show that the Killing vectors (2.14) transform according to

$$\boldsymbol{X}_i = \lambda_i{}^j \tilde{\boldsymbol{X}}_j, \qquad \boldsymbol{R}_{ij} = \lambda_i{}^k \lambda_j{}^\ell \tilde{\boldsymbol{R}}_{k\ell} + \mu_{ij}{}^k \tilde{\boldsymbol{X}}_k, \tag{2.15}$$

where

$$\mu_{ij}{}^k = \delta^\ell{}_{ijm} \lambda_\ell{}^k \delta^m. \tag{2.16}$$

The general Killing vector in $\mathcal{K}^1(\mathbb{E}^{n-s,s})$ may be expressed as

$$\boldsymbol{K} = A^i \boldsymbol{X}_i + B^{ij} \boldsymbol{R}_{ij}, \tag{2.17}$$

where the A^i and B^{ij} are constants and $B^{ij} = B^{[ij]}$. The action $\mathrm{SE}(n-s,s) \circlearrowright \mathcal{K}^1(\mathbb{E}^{n-s,s})$ follows immediately from the transformation rules (2.15) and reads

$$\tilde{A}^i = \lambda_j{}^i A^j + \mu_{jk}{}^i B^{jk}, \qquad \tilde{B}^{ij} = \lambda_k{}^i \lambda_\ell{}^j B^{k\ell}. \tag{2.18}$$

Extending to the valence-two case, the general Killing tensor in $\mathcal{K}^2(\mathbb{E}^{n-s,s})$ takes on the form

$$\boldsymbol{K} = A^{ij} \boldsymbol{X}_i \odot \boldsymbol{X}_j + 2B^{ijk} \boldsymbol{X}_i \odot \boldsymbol{R}_{jk} + C^{ijk\ell} \boldsymbol{R}_{ij} \odot \boldsymbol{R}_{k\ell}, \tag{2.19}$$

where the parameter objects A^{ij}, B^{ijk} and $C^{ijk\ell}$ satisfy the symmetries

$$A^{ij} = A^{(ij)}, \qquad B^{ijk} = B^{i[jk]}, \qquad C^{ijk\ell} = C^{k\ell ij} = C^{[ij][k\ell]}.$$

By (2.15), the action $\mathrm{SE}(n-s,s) \circlearrowright \mathcal{K}^2(\mathbb{E}^{n-s,s})$ is given by

$$\begin{aligned}
\tilde{A}^{ij} &= \lambda_k{}^i \lambda_\ell{}^j A^{k\ell} + 2\lambda_k{}^{(i} \mu_{\ell m}{}^{j)} B^{k\ell m} + \mu_{k\ell}{}^i \mu_{mn}{}^j C^{k\ell mn},\\
\tilde{B}^{ijk} &= \lambda_\ell{}^i \lambda_m{}^j \lambda_n{}^k B^{\ell mn} + \mu_{mn}{}^i \lambda_p{}^j \lambda_q{}^k C^{mnpq}\\
\tilde{C}^{ijk\ell} &= \lambda_m{}^i \lambda_n{}^j \lambda_p{}^k \lambda_q{}^\ell C^{mnpq}.
\end{aligned} \tag{2.20}$$

In order to streamline the generalization of the group actions (2.18) and (2.20) to arbitrary valence, it is convenient to introduce a multi-index notation. Henceforth, any upper-case index will represent two lower-case indices.

For example, $\boldsymbol{R}_I = \boldsymbol{R}_{ij}$ and $\mu_I{}^k = \mu_{ij}{}^k$. We also define

$$\lambda_M{}^J = \lambda_m{}^j \lambda_n{}^k.$$

The general Killing tensor in $\mathcal{K}^p(\mathbb{E}^{n-s,s})$ may be expressed as

$$\boldsymbol{K} = \sum_{q=0}^{p} \binom{p}{q} C_{\underline{p-q}}^{i_1 \cdots i_{p-q} J_{p-q+1} \cdots J_p} \boldsymbol{X}_{i_1} \odot \cdots \odot \boldsymbol{X}_{i_{p-q}} \odot \boldsymbol{R}_{J_{p-q+1}} \odot \cdots \odot \boldsymbol{R}_{J_p}. \tag{2.21}$$

The objects $C_{\underline{p-q}}^{i_1 \cdots i_{p-q} J_{p-q+1} \cdots J_p}$, for $q = 0, \ldots, p$, in (2.21) are constant and subject to the symmetries

$$C_{\underline{p-q}}^{i_1 \cdots i_{p-q} J_{p-q+1} \cdots J_p} = C_{\underline{p-q}}^{(i_1 \cdots i_{p-q})(J_{p-q+1} \cdots J_p)}.$$

The use of the underlined index $\underline{p-q}$ is to remind the reader that it is simply a label and not an index to be summed over. Using the Killing vector transformation rules (2.15) in conjunction with (2.21), it follows from an inductive argument that

$$\begin{aligned}\tilde{C}_{\underline{p-q}}^{i_1 \cdots i_{p-q} J_{p-q+1} \cdots J_p} &= \sum_{r=0}^{p-q} \binom{p-q}{r} \lambda_{\ell_1}{}^{(i_1} \cdots \lambda_{\ell_r}{}^{i_r} \mu_{M_{r+1}}{}^{i_{r+1}} \cdots \mu_{M_{p-q}}{}^{i_{p-q})} \\ &\quad \lambda_{M_{p-q+1}}{}^{J_{p-q+1}} \cdots \lambda_{M_p}{}^{J_p} C_{\underline{r}}^{\ell_1 \cdots \ell_r M_{r+1} \cdots M_p},\end{aligned} \tag{2.22}$$

for $q = 0, \ldots, p$. Therefore, we have proved the following theorem.

Theorem 3. *The group action* $\mathrm{SE}(n-s,s) \circlearrowright \mathcal{K}^p(\mathbb{E}^{n-s,s})$ *is given by Eq.* (2.22) *and is the explicit form of the representation of* $\mathrm{SE}(n-s,s)$ *on the vector space* $\mathcal{K}^p(\mathbb{E}^{n-s,s})$.

3. Computation of isometry group covariants and invariants

In this section, we present an algorithm concerning the generation of $\mathrm{SE}(n-s,s)$-invariants and covariants of $\mathcal{K}^p(\mathbb{E}^{n-s,s})$ using the group action derived in Section 2. In general, once an explicit form of the associated group action on a vector space is known, one can begin the search for invariants. In ITKT, two primary methods for computing invariants from a given group action have been employed previously, namely the *method of infinitesimal generators* and the *method of moving frames*. We now review these two methods and then give the main results of this section stated in Theorems 4 and 5.

At its heart, the method of infinitesimal generators is based on the fact that invariance of a function under the infinitesimal transformations of the group given by the corresponding Lie algebra is equivalent to invariance under the entire Lie group [22, Theorem 9.28]. Thus, in ITKT, any isometry group invariant is necessarily annihilated by the generators of $I(\mathcal{M})$ for the associated action $I(\mathcal{M}) \circlearrowright \mathcal{K}^p(\mathcal{M})$. This condition amounts to solving a system of (linear) partial differential equations (PDEs), the solutions of which are group invariants. Methods for solving these PDEs in ITKT are discussed in [5]. In summary, for isometry groups of sufficiently low dimension, the PDEs can be solved directly using the *method of characteristics* and their general solution is tractable in $\mathcal{K}^2(\mathbb{E}^2)$ [17,19,25] and $\mathcal{K}^2(\mathbb{M}^2)$ [18,20,25]. The generators of $\mathcal{K}^p(\mathbb{M}^2)$ are in fact known for arbitrary p [31], but a direct solution of the associated PDEs has not yet been attempted. When the method of characteristics fails, one can employ the more computational *method of undetermined coefficients*. Here, one forms a suitable polynomial ansatz for the invariants leading to a sparse system of linear equations for the undetermined coefficients, the solution of which is straightforward to implement in a computer algebra system. This method was successful for computing isometry group invariants of $\mathcal{K}^2(\mathbb{E}^3)$ [12] and $\mathcal{K}^3(\mathbb{E}^2)$ [14], the first examples in ITKT in which the invariants were presented in a compact indicial form. Evidently, the method of undetermined coefficients is dimensionally dependent and is not suitable for computing invariants of $\mathcal{K}^p(\mathbb{E}^{n-s,s})$ for arbitrary n and p.

The method of moving frames [7,8] is a purely algebraic method in which invariants are computed directly from the group action. The method relies on a suitable choice of *cross-section* through the group orbits. If the group acts regularly, the intersection of the cross-section with the orbits defines canonical forms of the space and their coordinates define a set of functionally independent group invariants. Moreover, the cross-section uniquely determines the associated *moving frame map* which sends any point on an orbit to its respective canonical form. Often, due to the complexity of the underlying group, an explicit construction of the moving frame map is not always possible or is exceedingly complicated. Although the full isometry group $\mathrm{SE}(n-s,s)$ is affected by these obstructions, it exhibits special topology which greatly simplifies the construction of the moving frame map and hence the generation of invariants. Recall that $\mathrm{SE}(n-s,s)$ is a semi-direct product of $\mathrm{SO}(n-s,s)$ and the group of translations in $\mathbb{E}^{n-s,s}$. Moreover, $\mathrm{SO}(n-s,s)$ is a closed subgroup of $\mathrm{SE}(n-s,s)$. Consequently, we can apply the recursive version of the method of moving frames, as developed by Kogan [15]. As we shall see, this modification of the classical method is extremely effective and drastically simplifies the computations. For the case of $\mathrm{SE}(n-s,s)$, the method of moving frames is applied in two steps. We first compute a set of fundamental invariants under the subgroup of translations. Then, using these translational invariants as new coordinates, we compute invariants under the action of the second subgroup, namely $\mathrm{SO}(n-s,s)$. These invariants are invariants of the full isometry group. Kogan's method was first employed in ITKT in [26] for $\mathcal{K}^2(\mathbb{M}^2)$ and later extended to $\mathcal{K}^p(\mathbb{M}^2)$ in [32]. The method is equally applicable in CIT. Most recently, group covariants were derived for general polynomial vector spaces, both inhomogeneous and homogeneous alike, independent of the degree of the polynomials and the number of variables [10]. We now extend the algorithm in [10] and give the analogous result in ITKT.

Theorem 4. *Consider the vector space $\mathcal{K}^p(\mathbb{E}^{n-s,s})$ whose general element is represented by (2.21). Define*

$$K_{\underline{p-q}}^{i_1\cdots i_{p-q}J_{p-q+1}\cdots J_p} = \sum_{r=0}^{p-q}\binom{p-q}{r}\delta^{(i_{r+1}}{}_{M_{r+1}\ell_{r+1}}\cdots\delta^{i_{p-q}}{}_{M_{p-q}\ell_{p-q}} C_{\underline{r}}^{i_1\cdots i_r)M_{r+1}\cdots M_{p-q}J_{p-q+1}\cdots J_p}x^{\ell_{r+1}}\cdots x^{\ell_{p-q}}, \tag{3.1}$$

for $q=0,\ldots,p$. Then, any scalar formed from contractions of the metric tensor g_{ij}, the Levi-Civita tensor $\epsilon_{i_1\cdots i_n}$ and $K_{\underline{p-q}}^{i_1\cdots i_{p-q}J_{p-q+1}\cdots J_p}$, $q=0,\ldots p$, is an $\mathrm{SE}(n-s,s)$-covariant of $\mathcal{K}^p(\mathbb{E}^{n-s,s})$.

Proof. We apply the recursive version of the moving frame method [15] and restrict the full group action $\mathrm{SE}(n-s,s) \circlearrowright (\mathcal{K}^p(\mathbb{E}^{n-s,s})\times\mathbb{E}^{n-s,s})$, given by Eqs. (2.9) and (2.22), to the subgroup of translations. This calculation amounts to substituting $\lambda_\ell{}^i=\delta^i{}_\ell$ and $\mu_M{}^j=\delta^j{}_{M\ell}\delta^\ell$ into (2.9) and (2.22). The cross-section we choose through the orbits is simply $\tilde{x}^i=0$, for $i=1,\ldots,n$. This choice of cross-section defines a *global* moving frame given by $\delta^i=x^i$. Substituting this map back into the restricted group action yields Eq. (3.1), where

$$K_{\underline{p-q}}^{i_1\cdots i_{p-q}J_{p-q+1}\cdots J_p}\equiv\tilde{C}_{\underline{p-q}}^{i_1\cdots i_{p-q}J_{p-q+1}\cdots J_p}.$$

Therefore, the $K_{\underline{p-q}}^{i_1\cdots i_{p-q}J_{p-q+1}\cdots J_p}$, $q=0,\ldots,p$, as defined in (3.1), constitute a set of fundamental translational covariants of $\mathcal{K}^p(\mathbb{E}^{n-s,s})$. We remark that the components $K^{i_1\cdots i_p}$ of the general Killing tensor in $\mathcal{K}^p(\mathbb{E}^{n-s,s})$ are given by $K_{\underline{p}}^{i_1\cdots i_p}$.

We now apply the second step of the recursive moving frame method using the translational covariants (3.1) as coordinates. Under the action of $\mathrm{SO}(n-s,s)$, the pseudo-Cartesian coordinates transform according to $x^i=\lambda^i{}_j\tilde{x}^j$ and it follows that

$$\tilde{K}_{\underline{p-q}}^{i_1\cdots i_{p-q}J_{p-q+1}\cdots J_p}=\lambda_{\ell_1}{}^{i_1}\cdots\lambda_{\ell_{p-q}}{}^{i_{p-q}}\lambda_{M_{p-q+1}}{}^{J_{p-q+1}}\cdots\lambda_{M_p}{}^{J_p}K_{\underline{p-q}}^{\ell_1\cdots\ell_{p-q}M_{p-q+1}\cdots M_p}, \tag{3.2}$$

for $q=0,\ldots,p$. Eq. (3.2) is a consequence of the fundamental identity

$$g_{ij}=\lambda^k{}_i\lambda^\ell{}_j g_{k\ell}. \tag{3.3}$$

Therefore, the translational covariants all transform like the components of a tensor under the action of $\mathrm{SO}(n-s,s)$. Indeed, it is not necessary to proceed with the moving frame method by defining a second cross-section. By the antisymmetry of $\epsilon_{i_1\cdots i_n}$ and the property $\det(\lambda^i{}_j)=1$, it follows that

$$\epsilon_{i_1\cdots i_n}=\lambda^{j_1}{}_{i_1}\cdots\lambda^{j_n}{}_{i_n}\epsilon_{j_1\cdots j_n}. \tag{3.4}$$

Thus, it is suffices to observe from (3.2)–(3.4) that any tensor product of the metric, the Levi-Civita tensor and the $K_{\underline{p-q}}^{i_1\cdots i_{p-q}J_{p-q+1}\cdots J_p}$, $q=0,\ldots,p$, is also a tensor. Thus, any contraction yielding a scalar from this tensor product is necessarily an $\mathrm{SE}(n-s,s)$-covariant of $\mathcal{K}^p(\mathbb{E}^{n-s,s})$. $\square$

A similar construction to that given in the proof of Theorem 4 exists for pure $\mathrm{SE}(n-s,s)$-invariants. However, we need to treat the cases $p=1$ and $p>1$ separately. For the vector space of Killing vectors $\mathcal{K}^1(\mathbb{E}^{n-s,s})$, the derivation of isometry group invariants is rather awkward because the dimension of generic orbits is strictly less than the dimension of the group, unlike the $p>1$ case where their dimensions coincide. Moreover, we need to treat the even- and odd-dimensional cases separately. Referring to Eq. (2.18), we see that the restriction of the action $\mathrm{SE}(n-s,s) \circlearrowright \mathcal{K}^1(\mathbb{E}^{n-s,s})$ to the subgroup of translations is simply

$$\tilde{A}^i = A^i + \delta^i{}_{jk\ell} B^{jk}\,\delta^\ell, \qquad \tilde{B}^{ij} = B^{ij}. \tag{3.5}$$

For $\mathcal{K}^1(\mathbb{E}^{2n-s,s})$, we can choose the cross-section $\tilde{A}^i=0$, for $i=1,\ldots,2n$; the resulting linear system of equations for the δ^i in (3.5) admits a unique solution for generic orbits and thus a (local) moving frame map exists. Therefore, all translational invariants depend only on the B^{ij} and hence any scalar formed from contractions of B^{ij}, g_{ij} and $\epsilon_{i_1\cdots i_{2n}}$ is an $\mathrm{SE}(2n-s,s)$-invariant of $\mathcal{K}^1(\mathbb{E}^{2n-s,s})$. In particular, the functions

$$\mathcal{I}_k = \mathrm{Tr}[(\boldsymbol{Bg})^{2k}], \tag{3.6}$$

for $k=1,\ldots,n$, form a set of functionally independent $\mathrm{SE}(2n-s,s)$-invariants of $\mathcal{K}^1(\mathbb{E}^{2n-s,s})$, where $\boldsymbol{Bg}$ is the matrix with components $(\boldsymbol{Bg})^i{}_j = B^{ik}g_{kj}$ and Tr is the trace operator. For $\mathcal{K}^1(\mathbb{E}^{2n+1-s,s})$, the situation is complicated due to the apparent absence of a ‘nice’ cross-section through the orbits. If we choose either one of the obvious cross-sections $\tilde{A}^i=0$ or $\delta^i{}_{jk\ell}\tilde{A}^j\tilde{B}^{k\ell}=0$, for $i=1,\ldots,2n+1$, it follows from (3.5) that the moving frame map exists, but is not unique. Indeed, the associated $(2n+1)\times(2n+1)$ coefficient matrix for the linear system has rank $2n$ almost everywhere. Consequently, in addition to the invariants (3.6), which also serve as invariants for the odd-dimensional case, we expect one additional invariant. It follows that

$$\mathcal{I}_{n+1} = \epsilon_{i_1 j_1\cdots i_n j_n k} B^{i_1 j_1}\cdots B^{i_n j_n} A^k \tag{3.7}$$

is an additional functionally independent $\mathrm{SE}(2n+1-s,s)$-invariant of $\mathcal{K}^1(\mathbb{E}^{2n+1-s,s})$.

Surprisingly, the derivation of isometry group invariants for $\mathcal{K}^p(\mathbb{E}^{n-s,s})$, when $p>1$, is not nearly as cumbersome as the $p=1$ case. Again, we apply our “hybrid” version of the recursive moving frame method [15]. We restrict the action $\mathrm{SE}(n-s,s) \circlearrowright \mathcal{K}^p(\mathbb{E}^{n-s,s})$ to the subgroup of translations. At the top level, we have

$$\tilde{C}_{\underline{0}}^{J_1\cdots J_p} = C_{\underline{0}}^{J_1\cdots J_p}, \qquad \tilde{C}_{\underline{1}}^{iJ_2\cdots J_p} = \delta^i{}_{M\ell} C_{\underline{0}}^{MJ_2\cdots J_p}\,\delta^\ell + C_{\underline{1}}^{iJ_2\cdots J_p}.$$

One choice of cross-section through the orbits is

$$\delta^i{}_{J_1\ell}\,\tilde{C}_{\underline{1}}{}^{\ell}{}_{J_2\cdots J_p}\,\tilde{C}_{\underline{0}}^{J_1\cdots J_p} = 0, \tag{3.8}$$

for $i=1,\ldots,n$. The resulting normalization equations for δ^i governing the moving frame map are of the form $A^i{}_k\delta^k = B^i$, where

$$\begin{aligned} A^i{}_k &= \delta^i{}_{J_1\ell}\,\delta^\ell{}_{Mk}\,C_{\underline{0}}{}^{M}{}_{J_2\cdots J_p} C_{\underline{0}}^{J_1\cdots J_p},\\ B^i &= -\delta^i{}_{J_1\ell}\,C_{\underline{1}}{}^{\ell}{}_{J_2\cdots J_p}\,C_{\underline{0}}^{J_1\cdots J_p}. \end{aligned} \tag{3.9}$$

For $p>1$, $|A|\equiv\det(A^i{}_k)\neq 0$ for generic orbits, hence a (local) moving frame map exists. Indeed, $\delta^i = (A^{-1})^i{}_k\,B^k = |A|^{-1}(\mathrm{adj}\,A)^i{}_k\,B^k$. Defining $\hat{\delta}^i\equiv|A|\delta^i$ and writing out the adjoint of A explicitly, we obtain

$$\hat{\delta}^i = \frac{(-1)^s}{(n-1)!}\epsilon_{ki_2\cdots i_n}\epsilon^{ij_2\cdots j_n}A^{i_2}{}_{j_2}\cdots A^{i_n}{}_{j_n}\,B^k. \tag{3.10}$$

Substituting (3.10) back into the restricted group action, we obtain the translational invariants

$$\begin{aligned} D_{\underline{p-q}}^{i_1\cdots i_{p-q}J_{p-q+1}\cdots J_p} = \sum_{r=0}^{p-q}\binom{p-q}{r}|A|^r\,\delta^{(i_{r+1}}{}_{M_{r+1}\ell_{r+1}}\cdots\delta^{i_{p-q}}{}_{M_{p-q}\ell_{p-q}}\\ C_{\underline{r}}^{i_1\cdots i_r)M_{r+1}\cdots M_{p-q}J_{p-q+1}\cdots J_p}\,\hat{\delta}^{\ell_{r+1}}\cdots\hat{\delta}^{\ell_{p-q}}, \end{aligned} \tag{3.11}$$

for $q = 0, \ldots, p$, where

$$D^{i_1 \cdots i_{p-q} J_{p-q+1} \cdots J_p}_{\underline{p-q}} \equiv |A|^{p-q} \tilde{C}^{i_1 \cdots i_{p-q} J_{p-q+1} \cdots J_p}_{\underline{p-q}}. \tag{3.12}$$

We remark that the translational invariants (3.11) are polynomials in the Killing tensor parameters. Since the $D^{i_1 \cdots i_{p-q} J_{p-q+1} \cdots J_p}_{\underline{p-q}}$ all transform like the components of a contravariant tensor, we have the following theorem.

Theorem 5. *Consider the vector space $\mathcal{K}^p(\mathbb{E}^{n-s,s})$ with $p > 1$ whose general element is represented by (2.21). Then, any scalar formed from contractions of $D^{i_1 \cdots i_{p-q} J_{p-q+1} \cdots J_p}_{\underline{p-q}}$, $q = 0, \ldots, p$, as defined in (3.11), the metric tensor g_{ij} and the Levi-Civita tensor $\epsilon_{i_1 \cdots i_n}$ is an $\mathrm{SE}(n-s,s)$-invariant of $\mathcal{K}^p(\mathbb{E}^{n-s,s})$.*

4. Invariant classification of separable webs in Euclidean space

The invariant theory of Killing tensors and the theory of separable webs can be cast elegantly into the framework of Cartan's geometry, as was shown by Adlam, McLenaghan and Smirnov [1]. We begin this section by reviewing the Cartan philosophy in the context of ITKT and then specialize to $\mathcal{K}^2(\mathbb{E}^3)$. Using Theorem 4, we then generate a set of fundamental isometry group covariants of $\mathcal{K}^2(\mathbb{E}^3)$ and use them to classify the eleven separable webs in $\mathbb{E}^3$. Our classification scheme is largely inspired by earlier work of Smirnov and Yue [25] who successfully employed group covariants to classify the separable webs in $\mathbb{E}^2$ and $\mathbb{M}^2$. We then conclude with a discussion of our results and their implications to other problems in ITKT.

The geometry of Cartan (see for example [9]) was first extended to ITKT in [1]. The study of the vector space of Killing tensors $\mathcal{K}^p(\mathbb{E}^{n-s,s})$ in Cartan's framework involves the following considerations. Firstly, we note that $\mathbb{E}^{n-s,s}$ defines a *homogeneous space*; the isometry group $\mathrm{SE}(n-s,s)$ acts transitively on $\mathbb{E}^{n-s,s}$ and $\mathrm{SO}(n-s,s)$ is the isotropy subgroup at each point $\boldsymbol{x} \in \mathbb{E}^{n-s,s}$. Thus, we have the principal fibre $\mathrm{SO}(n-s,s)$-bundle

$$\pi_1 : \mathrm{SE}(n-s,s) \to \mathrm{SE}(n-s,s)/\mathrm{SO}(n-s,s) \simeq \mathbb{E}^{n-s,s}.$$

Secondly, we can define two additional fibre bundles due to the vector space structure of $\mathcal{K}^p(\mathbb{E}^{n-s,s})$. The map

$$\pi_2 : \mathcal{K}^p(\mathbb{E}^{n-s,s}) \to \mathbb{E}^{n-s,s}$$

defines a vector bundle; the fibres are isomorphic to the vector space $\mathbb{R}^d$, where d is given by the dimension formula (2.3). Moreover, because the transitive action $\mathrm{SE}(n-s,s) \circlearrowright \mathbb{E}^{n-s,s}$ induces the corresponding non-transitive action $\mathrm{SE}(n-s,s) \circlearrowright \mathcal{K}^p(\mathbb{E}^{n-s,s})$, the study of the orbit space $\mathcal{K}^p(\mathbb{E}^{n-s,s})/\mathrm{SE}(n-s,s)$ plays a fundamental role in the solution of the equivalence problem. Thus, we have the structure of a principal fibre $\mathrm{SE}(n-s,s)$-bundle,

$$\pi_3 : \mathcal{K}^p(\mathbb{E}^{n-s,s}) \to \mathcal{K}^p(\mathbb{E}^{n-s,s})/\mathrm{SE}(n-s,s).$$

Finally, we can define a map $f : \mathcal{K}^p(\mathbb{E}^{n-s,s})/\mathrm{SE}(n-s,s) \to \mathrm{SE}(n-s,s)$ so that following diagram commutes:

$$\begin{tikzcd}
\mathrm{SE}(n-s,s) \arrow[r, "\pi_1"] & \mathrm{SE}(n-s,s)/\mathrm{SO}(n-s,s) \simeq \mathbb{E}^{n-s,s} \\
\mathcal{K}^p(\mathbb{E}^{n-s,s})/\mathrm{SE}(n-s,s) \arrow[u, "f"] & \mathcal{K}^p(\mathbb{E}^{n-s,s}) \arrow[l, "\pi_3"] \arrow[u, "\pi_2"']
\end{tikzcd} \tag{4.1}$$

The choice of a function f lifting the non-transitive action $\mathrm{SE}(n-s,s) \circlearrowright \mathcal{K}^p(\mathbb{E}^{n-s,s})$ to $\mathrm{SE}(n-s,s)$ is equivalent to choosing a cross-section through the orbits (or fixing the frame). The moving frame map corresponding to the cross-section prescribed by a chosen f is the composition $f \circ \pi_3 : \mathcal{K}^p(\mathbb{E}^{n-s,s}) \to \mathrm{SE}(n-s,s)$. The local invariants of the action $\mathrm{SE}(n-s,s) \circlearrowright \mathcal{K}^p(\mathbb{E}^{n-s,s})$ are the coordinates of the canonical forms obtained as the intersection of the orbits with the cross-section.

The case $p = 2$ and the study of the associated separable webs are intimately linked to the diagram (4.1).

Definition 6. Suppose $\boldsymbol{K} \in \mathcal{K}^2(\mathbb{E}^{n-s,s})$ has real and distinct eigenvalues and orthogonally integrable (normal) eigenvectors with respect to the metric tensor $\boldsymbol{g}$. The (orthogonally) separable web generated by $\boldsymbol{K}$ consists of n foliations of $\mathbb{E}^{n-s,s}$, the leaves of which are $n-1$-dimensional hypersurfaces orthogonal to the eigenvectors

Table 1
Canonical forms for the characteristic Killing tensors of $\mathcal{K}^2(\mathbb{E}^3)$

Separable web	Canonical characteristic Killing tensor
1 Cartesian	$A^{ij} = \mathrm{diag}(a_1, a_2, a_3)$, $B^{ij} = 0$, $C^{ij} = 0$
2 Circular cylindrical	$A^{ij} = \mathrm{diag}(a_1, a_1, a_3)$, $B^{ij} = 0$, $C^{ij} = \mathrm{diag}(0, 0, c_3)$
3 Parabolic cylindrical	$A^{ij} = \mathrm{diag}(a_1, a_1, a_3)$, $B^{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}$, $C^{ij} = 0$
4 Elliptic–hyperbolic	$A^{ij} = \mathrm{diag}(a_1, a_2, a_3)$, $B^{ij} = 0$, $C^{ij} = \mathrm{diag}(0, 0, c_3)$, $c_3(a_1 - a_2) > 0$
5 Spherical	$A^{ij} = \mathrm{diag}(a_1, a_1, a_1)$, $B^{ij} = 0$, $C^{ij} = \mathrm{diag}(c_2, c_2, c_3)$
6 Prolate spheroidal	$A^{ij} = \mathrm{diag}(a_1, a_1, a_3)$, $B^{ij} = 0$, $C^{ij} = \mathrm{diag}(c_2, c_2, c_3)$, $c_2(a_3 - a_1) > 0$
7 Oblate spheroidal	$A^{ij} = \mathrm{diag}(a_1, a_1, a_3)$, $B^{ij} = 0$, $C^{ij} = \mathrm{diag}(c_2, c_2, c_3)$, $c_2(a_3 - a_1) < 0$
8 Parabolic	$A^{ij} = \mathrm{diag}(a_1, a_1, a_1)$, $B^{ij} = \begin{pmatrix} 0 & b_{12} & 0 \\ -b_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $C^{ij} = \mathrm{diag}(0, 0, c_3)$
9 Conical	$A^{ij} = \mathrm{diag}(a_1, a_1, a_1)$, $B^{ij} = 0$, $C^{ij} = \mathrm{diag}(c_1, c_2, c_3)$
10 Paraboloidal	$A^{ij} = \mathrm{diag}(a_1, a_2, a_3)$, $B^{ij} = \begin{pmatrix} 0 & b_{12} & 0 \\ b_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $C^{ij} = \mathrm{diag}(0, 0, c_3)$,
	$b_{12}[b_{12}b_{21} + c_3(a_2 - a_3)] + b_{21}[b_{12}b_{21} + c_3(a_1 - a_3)] = 0$
11 Ellipsoidal	$A^{ij} = \mathrm{diag}(a_1, a_2, a_3)$, $B^{ij} = 0$, $C^{ij} = \mathrm{diag}(c_1, c_2, c_3)$, $(a_1 - a_2)c_1c_2 + (a_2 - a_3)c_2c_3 + (a_3 - a_1)c_3c_1 = 0$

of $\boldsymbol{K}$. Any valence-two Killing tensor which generates a separable web is called a characteristic Killing tensor (CKT).

Specializing the diagram (4.1) to CKTs defined in $\mathbb{E}^{n-s,s}$, we see that each fibre $\pi_1{}^{-1}(\boldsymbol{x})$ can be identified with an orthonormal frame of eigenvectors of $\boldsymbol{K}$. In this context, choosing the frame adapted to the normal eigenvectors of Killing tensors is equivalent to fixing the lift f and is exactly equivalent to defining cross-sections through the orbits which yield the canonical forms. The canonical forms of these CKTs characterize separable webs. The invariants are, by definition, the local coordinates of the canonical forms "sitting on" the orbits.

Each of the eleven separable webs in three-dimensional Euclidean space $\mathbb{E}^3$ is characterized by some canonical CKT (see [12] for their derivation). We list the eleven canonical CKTs of $\mathcal{K}^2(\mathbb{E}^3)$ in Table 1 in terms of the Killing tensor parameters A^{ij}, B^{ij} and C^{ij} defined in (2.8). Mimicking the proof of Theorem 4, we can generate SE(3)-covariants of $\mathcal{K}^2(\mathbb{E}^3)$. Indeed, we let

$$\begin{aligned} K^{ij} &= A^{ij} + 2\epsilon^{(i}{}_{\ell k}\, B^{j)k} x^\ell + \epsilon^i{}_{mk}\, \epsilon^j{}_{n\ell}\, C^{k\ell} x^m x^n, \\ L^{ij} &= B^{ij} + \epsilon^i{}_{\ell k} C^{jk}\, x^\ell. \end{aligned} \tag{4.2}$$

These quantities together with C^{ij} are the translational covariants of $\mathcal{K}^2(\mathbb{E}^3)$ which are obtained from the transformation rules (2.12) upon setting $\lambda^i{}_j = \delta^i{}_j$ and $\delta^i = x^i$. Therefore, any scalar formed from contractions of g_{ij}, ϵ_{ijk}, C^{ij}, K^{ij} and L^{ij} is an SE(3)-covariant of $\mathcal{K}^2(\mathbb{E}^3)$. By the fundamental theorem on invariants of a Lie group action [22, Theorem 8.17], the space $\mathcal{K}^2(\mathbb{E}^3) \times \mathbb{E}^3$ admits seventeen functionally independent SE(3)-covariants over which the group acts regularly. In our classification scheme of the eleven separable webs, we only require fifteen fundamental covariants. These covariants are given by

$$\begin{aligned} &\mathcal{C}_0 = \mathrm{Tr}(\boldsymbol{B}), \qquad \mathcal{C}_1 = \mathrm{Tr}(\boldsymbol{C}), \qquad \mathcal{C}_2 = \mathrm{Tr}(\boldsymbol{C}^2), \qquad \mathcal{C}_3 = \mathrm{Tr}(\boldsymbol{C}^3), \\ &\mathcal{C}_4 = \mathrm{Tr}(\boldsymbol{L}^2), \qquad \mathcal{C}_5 = \mathrm{Tr}(\boldsymbol{L}\boldsymbol{L}^t), \qquad \mathcal{C}_6 = \mathrm{Tr}(\boldsymbol{LCL}), \\ &\mathcal{C}_7 = \mathrm{Tr}(\boldsymbol{LCL}^t), \qquad \mathcal{C}_8 = \mathrm{Tr}(\boldsymbol{LC}^2\boldsymbol{L}), \qquad \mathcal{C}_9 = \mathrm{Tr}(\boldsymbol{K}), \\ &\mathcal{C}_{10} = \mathrm{Tr}(\boldsymbol{KC}), \qquad \mathcal{C}_{11} = \mathrm{Tr}(\boldsymbol{KC}^2), \qquad \mathcal{C}_{12} = \mathrm{Tr}(\boldsymbol{KC}^3), \\ &\mathcal{C}_{13} = \mathrm{Tr}(\boldsymbol{K}^2), \qquad \mathcal{C}_{14} = \mathrm{Tr}(\boldsymbol{K}^2\boldsymbol{C}), \qquad \mathcal{C}_{15} = \mathrm{Tr}(\boldsymbol{KCKC}). \end{aligned} \tag{4.3}$$

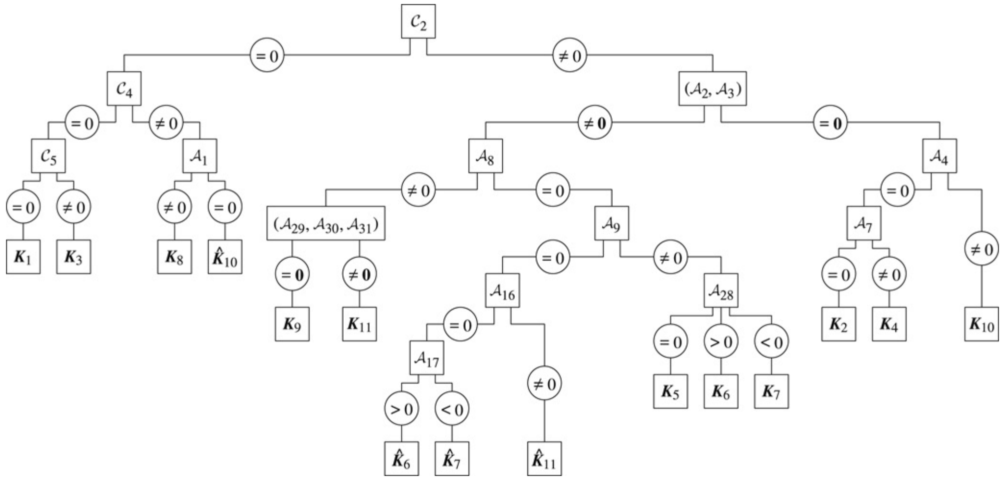

Fig. 1. Classification of the separable webs in three-dimensional Euclidean space.

In Eq. (4.3), the covariants $\mathcal{C}_0, \dots, \mathcal{C}_{15}$ are written in terms of traces of the matrices $\boldsymbol{C}$, $\boldsymbol{K}$ and $\boldsymbol{L}$ with components $(\boldsymbol{L})^i{}_j = L^i{}_j = L^{ik} g_{kj}$, $(\boldsymbol{L}^t)^i{}_j = L_j{}^i = g_{jk} L^{ki}$, etc. The covariant $\mathcal{C}_0$ is not a 'true' covariant per se, but rather a constraint (which may be set to zero) on account of writing the general Killing tensor in (2.8) using twenty-one parameters, as opposed to twenty, the actual dimension of $\mathcal{K}^2(\mathbb{E}^3)$.

The process of generating other covariants and pure invariants from a given covariant is called *transvection* (see [22, Chapter 5] for its applicability in CIT). For the problem of distinguishing between the orbits of the eleven classes of CKTs in Table 1, the generation of pure invariants from one or more covariants plays a crucial role in the classification problem. Eqs. (4.3) serve as invariants of the *extended space* $\mathcal{K}^2(\mathbb{E}^3) \times \mathbb{E}^3$, yet we are essentially interested in separating orbits of the *unprolonged space* $\mathcal{K}^2(\mathbb{E}^3)$. In some instances, certain polynomial combinations of the fundamental covariants produce pure invariants. Often, these combinations can be found by inspection, especially if one is attempting to distinguish between two given CKTs. Another technique, motivated from the "transvection" process, is to apply certain invariant differential operators to a covariant $\mathcal{C}$. Indeed, under the isometry group SE(3), the partial derivative operator transforms like $\tilde{\partial}_i = \lambda^j{}_i \partial_j$. Therefore, the Laplacian $\nabla^2 \mathcal{C} = g^{ij} \partial_i \partial_j \mathcal{C}$ is also a covariant; if $\mathcal{C}$ is at most quadratic in the Cartesian coordinates x^i, then $\nabla^2 \mathcal{C}$ is a pure invariant. Similarly, $|\nabla \mathcal{C}|^2 = g^{ij}(\partial_i \mathcal{C})(\partial_j \mathcal{C})$ is a covariant. Moreover, if $\mathcal{C}$ and $\mathcal{D}$ are covariants, the dot product of their gradients, $\nabla \mathcal{C} \cdot \nabla \mathcal{D} = g^{ij}(\partial_i \mathcal{C})(\partial_j \mathcal{D})$ is also a covariant. In our development of a classification scheme for the separable webs of $\mathbb{E}^3$, we will need to define many auxiliary covariants or transvectants from the fundamental set (4.3). These covariants are listed in the proof of Proposition 7 which we now state.

Proposition 7. *Let $\boldsymbol{K}_1, \dots, \boldsymbol{K}_{11}$ denote the eleven canonical CKTs of $\mathcal{K}^2(\mathbb{E}^3)$ listed in Table 1. An invariant classification of $\boldsymbol{K}_1, \dots, \boldsymbol{K}_{11}$ is described by the flowchart in Fig. 1.*

Proof. It is necessary to define certain "degenerate" cases for some of the canonical CKTs. That said, we define

$$\hat{\boldsymbol{K}}_6 = \boldsymbol{K}_6|_{c_3=c_2}, \qquad \hat{\boldsymbol{K}}_7 = \boldsymbol{K}_7|_{c_3=c_2}, \qquad \hat{\boldsymbol{K}}_{10} = \boldsymbol{K}_{10}|_{c_3=0,\, b_{21}=-b_{12}}, \qquad \hat{\boldsymbol{K}}_{11} = \boldsymbol{K}_{11}|_{c_2=c_1,\, c_3=c_1}. \tag{4.4}$$

Note that the "unhatted" Killing tensors in Table 1 are just the complement of the set of "hatted" tensors defined in (4.4). For example, the CKT $\boldsymbol{K}_6$ satisfies $c_3 \neq c_2$. Note too that an unhatted CKT and its hatted counterpart still define the same type of separable web.

To begin, we first evaluate the covariant $\mathcal{C}_2$ on each of the CKTs in (4.4) and Table 1. As $\mathcal{C}_2$ is just a sum of squares of the c_i parameters, it follows that $\mathcal{C}_2$ vanishes only for $\boldsymbol{K}_1, \boldsymbol{K}_3, \boldsymbol{K}_8$ and $\hat{\boldsymbol{K}}_{10}$. We now split the analysis into two cases: $\mathcal{C}_2 = 0$ and $\mathcal{C}_2 \neq 0$.

Case I. Suppose $\mathcal{C}_2 = 0$. If we evaluate $\mathcal{C}_4$, then it vanishes for $\boldsymbol{K}_1$ and $\boldsymbol{K}_3$, while for $\boldsymbol{K}_8$ and $\hat{\boldsymbol{K}}_{10}$, $\mathcal{C}_4 = -2b_{12}{}^2 \neq 0$. To distinguish between $\boldsymbol{K}_1$ and $\boldsymbol{K}_3$, we note that $\mathcal{C}_5$ vanishes for the former, while for the latter, $\mathcal{C}_5 = b_{23}{}^2 \neq 0$. To distinguish between $\boldsymbol{K}_8$ and $\hat{\boldsymbol{K}}_{10}$, we define an auxiliary covariant

$$\mathcal{A}_1 = \nabla^2 \mathcal{C}_5.$$

Then, $\mathcal{A}_1 = 0$ for $\hat{\boldsymbol{K}}_{10}$ and $\mathcal{A}_1 = 4{c_3}^2 \neq 0$ for $\boldsymbol{K}_8$.

Case II. Suppose $\mathcal{C}_2 \neq 0$. We define two additional auxiliary covariants

$$\mathcal{A}_2 = \mathcal{C}_2 - {\mathcal{C}_1}^2, \qquad \mathcal{A}_3 = \mathcal{C}_3 - {\mathcal{C}_1}^3 .$$

Indeed, $\mathcal{A}_2 = \mathcal{A}_3 = 0$ for $\boldsymbol{K}_2, \boldsymbol{K}_4$ and $\boldsymbol{K}_{10}$. For the remaining CKTs, we have the following evaluations:

$$\begin{aligned}
\boldsymbol{K}_5,\ \boldsymbol{K}_6,\ \boldsymbol{K}_7 &: \quad \mathcal{A}_2 = -2c_2(c_2 + 2c_3), \qquad \mathcal{A}_3 = -6c_2(c_2 + c_3)^2 \\
\hat{\boldsymbol{K}}_6,\ \hat{\boldsymbol{K}}_7 &: \quad \mathcal{A}_2 = -6{c_2}^2, \qquad \mathcal{A}_3 = -24{c_2}^3 \\
\boldsymbol{K}_9,\ \boldsymbol{K}_{11} &: \quad \mathcal{A}_2 = -2(c_1c_2 + c_2c_3 + c_3c_1), \qquad \mathcal{A}_3 = -3(c_1 + c_2)(c_2 + c_3)(c_3 + c_1) \\
\hat{\boldsymbol{K}}_{11} &: \quad \mathcal{A}_2 = -6{c_1}^2, \qquad \mathcal{A}_3 = -24{c_1}^3 .
\end{aligned}$$

For these cases, $\mathcal{A}_2$ and $\mathcal{A}_3$ cannot both vanish identically, otherwise the CKTs would either fail to have distinct eigenvalues or satisfy $\mathcal{C}_2 = 0$. There are now two subcases to consider.

Case II.1. Suppose $\mathcal{A}_2 = \mathcal{A}_3 = 0$. To distinguish between the CKTs $\boldsymbol{K}_2, \boldsymbol{K}_4$ and $\boldsymbol{K}_{10}$ of this case, we define

$$\begin{aligned}
&\mathcal{A}_4 = \mathcal{C}_1\mathcal{C}_5 - \mathcal{C}_7, \qquad \mathcal{A}_5 = {\mathcal{C}_1}^2(\mathcal{C}_1\mathcal{C}_9 - \mathcal{C}_5 - \mathcal{C}_{10}), \\
&\mathcal{A}_6 = {\mathcal{C}_1}^2\,\mathcal{C}_{13} - {\mathcal{C}_5}^2 - {\mathcal{C}_{10}}^2, \qquad \mathcal{A}_7 = \nabla^2|\nabla\mathcal{A}_6|^2 - 16{\mathcal{A}_5}^2 .
\end{aligned}$$

It follows that $\mathcal{A}_4 = 0$ for $\boldsymbol{K}_2$ and $\boldsymbol{K}_4$, while for $\boldsymbol{K}_{10}$, $\mathcal{A}_4 = c_3({b_{12}}^2 + {b_{21}}^2) \neq 0$. Finally, $\mathcal{A}_7 = 16{c_3}^6(a_1 - a_2)^2 \neq 0$ for $\boldsymbol{K}_4$ and vanishes identically for $\boldsymbol{K}_2$.

Case II.2. Suppose $(\mathcal{A}_2, \mathcal{A}_3) \neq \mathbf{0}$. We define the auxiliary covariant

$$\mathcal{A}_8 = {\mathcal{C}_1}^2 - 3{\mathcal{C}_2}^3 + 18{\mathcal{C}_3}^2 - 3{\mathcal{C}_1}^2\mathcal{C}_2(3{\mathcal{C}_1}^2 - 7\mathcal{C}_2) + 4\mathcal{C}_1\mathcal{C}_3(2{\mathcal{C}_1}^2 - 9\mathcal{C}_2).$$

This covariant vanishes for $\boldsymbol{K}_5, \boldsymbol{K}_6, \boldsymbol{K}_7, \hat{\boldsymbol{K}}_6, \hat{\boldsymbol{K}}_7$ and $\hat{\boldsymbol{K}}_{11}$, while for $\boldsymbol{K}_9$ and $\boldsymbol{K}_{11}$,

$$\mathcal{A}_8 = -6(c_1 - c_2)^2(c_2 - c_3)^2(c_3 - c_1)^2 \neq 0.$$

We now split the analysis into two further subcases: $\mathcal{A}_8 = 0$ and $\mathcal{A}_8 \neq 0$.

Case II.2.i. Suppose $\mathcal{A}_8 = 0$ and define

$$\mathcal{A}_9 = {\mathcal{C}_1}^2 - 3\mathcal{C}_2.$$

The auxiliary covariant $\mathcal{A}_9$ vanishes identically for the CKTs $\hat{\boldsymbol{K}}_6, \hat{\boldsymbol{K}}_7$ and $\hat{\boldsymbol{K}}_{11}$, while for $\boldsymbol{K}_5, \boldsymbol{K}_6$ and $\boldsymbol{K}_7$, $\mathcal{A}_9 = -2(c_2 - c_3)^2 \neq 0$. There are two final subcases to consider.

Case II.2.i.a. Suppose $\mathcal{A}_8 = 0$ and $\mathcal{A}_9 = 0$. We define a series of auxiliary covariants given by

$$\begin{aligned}
&\mathcal{A}_{10} = {\mathcal{C}_1}^2(\mathcal{C}_4 + \mathcal{C}_{10}), \qquad \mathcal{A}_{11} = 2{\mathcal{C}_1}^2\,\mathcal{C}_{13} - 9{\mathcal{C}_4}^2, \qquad \mathcal{A}_{12} = |\nabla\mathcal{A}_{11}|^2, \\
&\mathcal{A}_{13} = \nabla^2\mathcal{A}_{12}, \qquad \mathcal{A}_{14} = \nabla^4(\mathcal{A}_{11}\mathcal{A}_{12}), \qquad \mathcal{A}_{15} = \nabla^2(\nabla\mathcal{A}_{11}\cdot\nabla\mathcal{A}_{12}), \\
&\mathcal{A}_{16} = 2^{27}{\mathcal{A}_{10}}^6 - 2^{18}\cdot 3^2{\mathcal{A}_{10}}^4\,\mathcal{A}_{13} + 2^{16}{\mathcal{A}_{10}}^3\,\mathcal{A}_{15} + 2^9\cdot 3\cdot 7{\mathcal{A}_{10}}^2\,{\mathcal{A}_{13}}^2 \\
&\qquad\quad - 2^6\cdot 3^2\mathcal{A}_{10}\mathcal{A}_{13}\mathcal{A}_{15} - 3{\mathcal{A}_{13}}^3 + 3^2{\mathcal{A}_{15}}^2, \\
&\mathcal{A}_{17} = 2^{15}{\mathcal{A}_{10}}^3 - 9\mathcal{A}_{14} + 27\mathcal{A}_{15}.
\end{aligned}$$

For $\hat{\boldsymbol{K}}_{11}$, it follows that

$$\mathcal{A}_{16} = -2^{22}\cdot 3^{13}{c_1}^{18}(a_1 - a_2)^2(a_2 - a_3)^2(a_3 - a_1)^2 \neq 0,$$

while for $\hat{\boldsymbol{K}}_6$ and $\hat{\boldsymbol{K}}_7$, $\mathcal{A}_{16} = 0$. However, for these two CKTs, $\mathcal{A}_{17} = {c_2}^9(a_3 - a_1)^3$, which is positive for $\hat{\boldsymbol{K}}_6$ and negative for $\hat{\boldsymbol{K}}_7$.

Case II.2.i.b. Suppose $\mathcal{A}_8 = 0$ and $\mathcal{A}_9 \neq 0$. The following auxiliary covariants are required for this subcase:

$$\begin{aligned}
&\mathcal{A}_{18} = {\mathcal{C}_1}^2(\mathcal{C}_4 + \mathcal{C}_{10}), \qquad \mathcal{A}_{19} = \mathcal{C}_2(\mathcal{C}_4 + \mathcal{C}_{10}), \qquad \mathcal{A}_{20} = \mathcal{C}_1(\mathcal{C}_6 + \mathcal{C}_{11}), \\
&\mathcal{A}_{21} = \mathcal{C}_8 + \mathcal{C}_{12}, \qquad \mathcal{A}_{22} = \mathcal{C}_1(\mathcal{C}_4\mathcal{C}_9 + 2\mathcal{C}_{14}), \qquad \mathcal{A}_{23} = {\mathcal{C}_4}^2 - 2\mathcal{C}_{15},
\end{aligned}$$

$$
\begin{aligned}
&\mathcal{A}_{24} = \nabla^2\mathcal{A}_{22}, \qquad \mathcal{A}_{25} = \nabla\mathcal{A}_{22}\cdot\nabla(\nabla^2\mathcal{A}_{22}),\\
&\mathcal{A}_{26} = \nabla\mathcal{A}_{23}\cdot\nabla(\nabla^2\mathcal{A}_{23}), \qquad \mathcal{A}_{27} = \det(\nabla\nabla\mathcal{A}_{23}),\\
&\mathcal{A}_{28} = 28304{\mathcal{A}_{18}}^2\mathcal{A}_{19} - 52672{\mathcal{A}_{18}}^2\mathcal{A}_{20} + 43008{\mathcal{A}_{18}}^2\mathcal{A}_{21} - 26464\mathcal{A}_{18}{\mathcal{A}_{19}}^2\\
&\quad - 89664\mathcal{A}_{18}\mathcal{A}_{19}\mathcal{A}_{20} + 113696\mathcal{A}_{18}\mathcal{A}_{19}\mathcal{A}_{21} + 261728\mathcal{A}_{18}{\mathcal{A}_{20}}^2\\
&\quad - 491616\mathcal{A}_{18}\mathcal{A}_{20}\mathcal{A}_{21} + 204448\mathcal{A}_{18}{\mathcal{A}_{21}}^2 + 1024\mathcal{A}_{18}{\mathcal{A}_{24}}^2 - 1840{\mathcal{A}_{19}}^3\\
&\quad - 9760{\mathcal{A}_{19}}^2\mathcal{A}_{20} - 4608{\mathcal{A}_{19}}^2\mathcal{A}_{21} + 7880{\mathcal{A}_{19}}^2\mathcal{A}_{24} + 72000\mathcal{A}_{19}{\mathcal{A}_{20}}^2\\
&\quad - 12768\mathcal{A}_{19}\mathcal{A}_{20}\mathcal{A}_{21} - 33792\mathcal{A}_{19}{\mathcal{A}_{21}}^2 - 15328\mathcal{A}_{19}\mathcal{A}_{21}\mathcal{A}_{24}\\
&\quad - 2012\mathcal{A}_{19}{\mathcal{A}_{24}}^2 - 90912{\mathcal{A}_{20}}^3 + 26464{\mathcal{A}_{20}}^2\mathcal{A}_{21} - 40232{\mathcal{A}_{20}}^2\mathcal{A}_{24}\\
&\quad + 150976\mathcal{A}_{20}{\mathcal{A}_{21}}^2 + 102528\mathcal{A}_{20}\mathcal{A}_{21}\mathcal{A}_{24} - 86528{\mathcal{A}_{21}}^3\\
&\quad - 45088{\mathcal{A}_{21}}^2\mathcal{A}_{24} - 72\mathcal{A}_{21}{\mathcal{A}_{24}}^2 - 2{\mathcal{A}_{24}}^3 + 792\mathcal{A}_{19}\mathcal{A}_{26}\\
&\quad - 1584\mathcal{A}_{20}\mathcal{A}_{26} + 24\mathcal{A}_{21}\mathcal{A}_{25} + 2016\mathcal{A}_{21}\mathcal{A}_{26} + 27\mathcal{A}_{27}.
\end{aligned}
$$

The complicated combination $\mathcal{A}_{28}$ has been constructed so that it evaluates to $1024{c_2}^9(a_3 - a_1)^3$ for $\boldsymbol{K}_5$, $\boldsymbol{K}_6$ and $\boldsymbol{K}_7$, which is zero, positive and negative on these CKTs, respectively.

Case II.2.ii. Suppose $\mathcal{A}_8 \neq 0$. To distinguish between $\boldsymbol{K}_9$ and $\boldsymbol{K}_{11}$, we define the auxiliary covariants

$$
\begin{aligned}
&\mathcal{A}_{29} = \mathcal{C}_1\mathcal{C}_2(\mathcal{C}_4 + \mathcal{C}_{10}) - {\mathcal{C}_1}^2(\mathcal{C}_6 + \mathcal{C}_{11}),\\
&\mathcal{A}_{30} = \mathcal{C}_3(\mathcal{C}_4 + \mathcal{C}_{10}) - \mathcal{C}_1(\mathcal{C}_8 + \mathcal{C}_{12}),\\
&\mathcal{A}_{31} = 2\mathcal{C}_2(\mathcal{C}_6 + \mathcal{C}_{11}) - 2\mathcal{C}_1\mathcal{C}_2(\mathcal{C}_4 + \mathcal{C}_{10}) + \mathcal{C}_2\nabla^2(\mathcal{C}_4\mathcal{C}_9 + 2\mathcal{C}_{14}).
\end{aligned}
$$

It follows that $\mathcal{A}_{29}$, $\mathcal{A}_{30}$ and $\mathcal{A}_{31}$ vanish identically for $\boldsymbol{K}_9$, while for $\boldsymbol{K}_{11}$,

$$
\begin{aligned}
&\mathcal{A}_{29} = (c_1 + c_2 + c_3)[c_1c_2(c_1 - c_2)(a_2 - a_1) + c_2c_3(c_2 - c_3)(a_3 - a_2) + c_3c_1(c_3 - c_1)(a_1 - a_3)],\\
&\mathcal{A}_{30} = c_1c_2({c_1}^2 - {c_2}^2)(a_2 - a_1) + c_2c_3({c_2}^2 - {c_3}^2)(a_3 - a_2) + c_3c_1({c_3}^2 - {c_1}^2)(a_1 - a_3),\\
&\mathcal{A}_{31} = 2({c_1}^2 + {c_2}^2 + {c_3}^2)[c_1c_2(a_1 + a_2 - 2a_3) + c_2c_3(a_2 + a_3 - 2a_1) + c_3c_1(a_3 + a_1 - 2a_2)].
\end{aligned}
$$

These three covariants cannot simultaneously vanish for $\boldsymbol{K}_{11}$. Indeed, if all three were to vanish, then the Killing tensor parameters would necessarily satisfy one of the following five cases:

$$
c_1 = c_2 = 0, \qquad c_2 = c_3 = 0, \qquad c_3 = c_1 = 0, \qquad a_1 = a_2 = a_3, \qquad c_1 = c_2 = c_3.
$$

Clearly, all fives cases are impossible, since the CKT $\boldsymbol{K}_{11}$ would either reduce to $\boldsymbol{K}_9$ or contradict the condition $\mathcal{A}_8 \neq 0$. This completes the proof of the proposition for the derivation of the classification scheme of the canonical CKTs characterizing the eleven separable webs in $\mathbb{E}^3$. □

Our method for classifying the eleven orthogonally separable webs in $\mathbb{E}^3$ highlights the importance of the use of covariants in ITKT. Our classification scheme depicted in Fig. 1 uses the fundamental covariants (4.3) and suitable transvectants. The approach used in this paper should be contrasted with that given in [12] which employed both invariants and reduced invariants. Moreover, it was observed in [12] that invariants alone were insufficient to distinguish between all eleven webs. The reason for these shortcomings of invariants is simple; they are derived using *local* methods and hence at best can only provide a *local* characterization of the orbits. Indeed, any functionally independent set of $\mathrm{SE}(n-s,s)$-invariants of $\mathcal{K}^p(\mathbb{E}^{n-s,s})$ constructed using Theorem 5 are only guaranteed to distinguish between those Killing tensors satisfying a certain determinant condition (see Eq. (3.9)). For $\mathcal{K}^2(\mathbb{E}^3)$ in particular, a natural choice for the cross-section is $\epsilon_{ijk}\tilde{B}^{jk} = 0$ which leads to the condition $\det(C^i{}_j - C^k{}_k\,\delta^i{}_j) \neq 0$. It is satisfied only for the spherical, oblate spheroidal, prolate spheroidal, conical and ellipsoidal CKTs. Thus, any set of fundamental invariants constructed from this cross-section will fail to discriminate between the other six separable webs.

In contrast to pure isometry group invariants, the covariants constructed through the algorithm of this paper have the desired "discriminating power", largely as a consequence of prolongation. As was observed in [3], the deficiencies in a group action can often be removed by extending the space (e.g. taking a Cartesian product of the space with itself) on which it acts. In the context of ITKT, a natural prolongation is the extension of the vector space $\mathcal{K}^p(\mathcal{M})$ to the product space $\mathcal{K}^p(\mathcal{M}) \times \mathcal{M}$. By definition, group invariants of the prolonged space are group covariants of the unprolonged space. Reflecting on the proof of Theorem 4, we observe that all orbits of the prolonged space $\mathcal{K}^p(\mathbb{E}^{n-s,s}) \times \mathbb{E}^{n-s,s}$, under the subgroup of translations, have maximal dimension n. The unprolonged space $\mathcal{K}^p(\mathbb{E}^{n-s,s})$ does not enjoy this property. We also note that the cross-section $\tilde{x}^i = 0$ intersects the orbits transversally and yields a *global* moving frame map. The existence of this global cross-section can be explained beautifully using the fibre bundle theory. Indeed, we have a principal bundle with the fibration

$$\pi : \mathcal{K}^p(\mathbb{E}^{n-s,s}) \times \mathbb{E}^{n-s,s} \to (\mathcal{K}^p(\mathbb{E}^{n-s,s}) \times \mathbb{E}^{n-s,s})/G,$$

where G is the subgroup of translations. As G is identifiable with $\mathbb{E}^{n-s,s}$, each fibre of this principal bundle is G itself, isomorphic to $\mathbb{E}^{n-s,s}$. Therefore, the principal G-bundle is *trivial* and hence, by a result in the fibre bundle theory [27], is equivalent to the admittance of a global cross-section. We thank Roman Smirnov for clarifying these points [24].

Our algorithm for generating isometry group invariants and covariants of vector spaces of Killing tensors is independent of the dimension or signature of the underlying flat manifold and the valence of the tensors. As a consequence of prolongation, the use of covariants provides a practical tool for characterizing group orbits. Thus, the classification of separable webs or group orbits for any space of Killing tensors defined on a pseudo-Riemannian manifold of constant curvature is amenable to the methods developed in this work.

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